# Appendix A

# Math Reviews 03Jan2007

**Objectives**

1. Review tools that are needed for studying models for CLDVs.
2. Get you used to the notation that will be used.

**Readings**

1. Read this appendix before class.
2. Pay special attention to the results marked with a *.
3. Review any other algebra text as needed.

## A.1 From Simple to Complex

▶ With a simple equation:

$$x = y$$

▶ Or a complex equation:

$$y = b_0 + b_1 x_1 + b_2 x_2 + \cdots + u$$

▶ The same rules apply. Don't confuse messy and complex with hard and incomprehensible!

## A.2 Basic Rules

**Distributive law**

$$\begin{aligned} a \times (b + c) &= (a \times b) + (a \times c) \qquad \text{(A.1)} \\ 4 \times (2 + 3) &= (4 \times 2) + (4 \times 3) \end{aligned}$$

$$\begin{aligned} (\phi_1 - \phi_2)(\beta_0 + \beta_1 x_1 + \beta_2 x_2) &= (\phi_1 - \phi_2)\,\Delta \qquad \text{(A.2)} \\ &= \phi_1\Delta - \phi_2\Delta \\ &= \phi_1(\beta_0 + \beta_1 x_1 + \beta_2 x_2) - \phi_2(\beta_0 + \beta_1 x_1 + \beta_2 x_2) \\ &= [\phi_1\beta_0 + \phi_1\beta_1 x_1 + \phi_1\beta_2 x_2] - [\phi_2\beta_0 + \phi_2\beta_1 x_1 + \phi_2\beta_2 x_2] \end{aligned}$$

**Multiplying by** $1$

$$\frac{a}{b} = 1 \times \frac{a}{b} = \frac{k}{k} \times \frac{a}{b} = \frac{ka}{kb} \qquad \text{(A.3)}$$

$$\frac{2}{3} = 1 \times \frac{2}{3} = \frac{4}{4} \times \frac{2}{3} = \frac{4 \times 2}{4 \times 3} = \frac{8}{12} = \frac{2}{3}$$

## A.3 Solving Equations

Let $p$ be the probability of an event, and $\Omega = \frac{p}{1-p}$ the odds (*Note*: $\Omega = e^{x\beta}$). You should be able to work this derivation from $\Omega$ to $p$ and from $p$ to $\Omega$ without looking.

$$\Omega = \frac{p}{1-p} \qquad 9 = \frac{.9}{.1}$$

$$\Omega(1-p) = p \qquad 9(1-.9) = .9$$

$$\Omega - \Omega p = p \qquad 9 - 9(.9) = .9$$

$$\Omega = \Omega p + p \qquad 9 = 9(.9) + .9$$

$$\Omega = p(1+\Omega) \qquad 9 = .9(1+9)$$

$$\frac{\Omega}{1+\Omega} = p \qquad \frac{9}{1+9} = .9$$

Therefore, $p = \frac{\Omega}{1+\Omega} = \frac{e^{x\beta}}{1+e^{x\beta}}$.

## A.4 Exponents and Radicals

**Zero exponent**

$$\begin{aligned} a^0 &= 1 \qquad \text{(A.4)} \\ 3^0 &= 1 \\ 2.718128^0 &= 1 = e^0 \end{aligned}$$

**Integer exponent**

$$\begin{aligned} a^k &= a \cdots (k) \cdots a, \text{ where } (k) \text{ means } \textit{repeat } k \textit{ times} \qquad \text{(A.5)} \\ 2^3 &= 2 \times 2 \times 2 = 8 \\ e^3 &= 2.71828 \times 2.71828 \times 2.71828 = 20.086 \end{aligned}$$

**Negative integer exponent**

$$\begin{aligned} a^{-k} &= \frac{1}{a \cdots (k) \cdots a} = \frac{1}{a^k} \qquad \text{(A.6)} \\ 2^{-3} &= \frac{1}{2 \times 2 \times 2} = \frac{1}{8} \end{aligned}$$

**Base e** $e = 2.71828182846\ldots$ is a useful base. Notation is: $e^x$ or $\exp(x)$.

$$e^0 = 1 \qquad e^1 = 2.718 \qquad e^2 = e \times e = 7.389 \qquad e^3 = e \times e \times e = 20.086$$

*** Product of powers: multiplying as the sum of powers**

$$\begin{aligned} a^M a^N &= [a \cdots (M+N) \cdots a] = a^{M+N} \qquad \text{(A.7)} \\ 2^3 2^4 &= (2 \times 2 \times 2)\,(2 \times 2 \times 2 \times 2) = 2^{3+4} = 2^7 \\ e^3 e^4 &= (e \times e \times e)\,(e \times e \times e \times e) = e^{3+4} = e^7 \qquad \text{(A.8)} \end{aligned}$$

*** Quotient of powers**

$$\begin{aligned} \frac{a^M}{a^N} &= \frac{[a \cdots (M) \cdots a]}{[a \cdots (N) \cdots a]} = a^{M-N} \qquad \text{(A.9)} \\ \frac{e^5}{e^3} &= \frac{e \times e \times e \times e \times e}{e \times e \times e} = e^{5-3} = e^2 \end{aligned}$$

**Power of powers**

$$\begin{aligned}(a^M)^N &= a^{MN} \\ \left(e^2\right)^5 &= (e \times e)\,(e \times e)\,(e \times e)\,(e \times e)\,(e \times e) = e^{10} = e^{2\times 5}\end{aligned} \tag{A.10}$$

## A.5 ** Natural Logarithms

Natural logarithms and exponentials are used extensively in statistics. A key reason is that they turn multiplication into addition. Here's why:

1. Every positive real number $m$ can be written as

$$m = e^{\,p}$$

2. Example: Let $m = 13$. Find $p$ such that $e^p = 13$.

   (a) $e^2 = 7.389\ldots$ and $e^3 = 20.086\ldots \Rightarrow\ 2 < p < 3$.

   (b) $e^{2.5} = 12.128\ldots$ and $e^{2.6} = 13.464\ldots \Rightarrow\ 2.5 < p < 2.6$.

   (c) And so on until $e^{2.565\ldots} = 13$

3. Definition of the Log

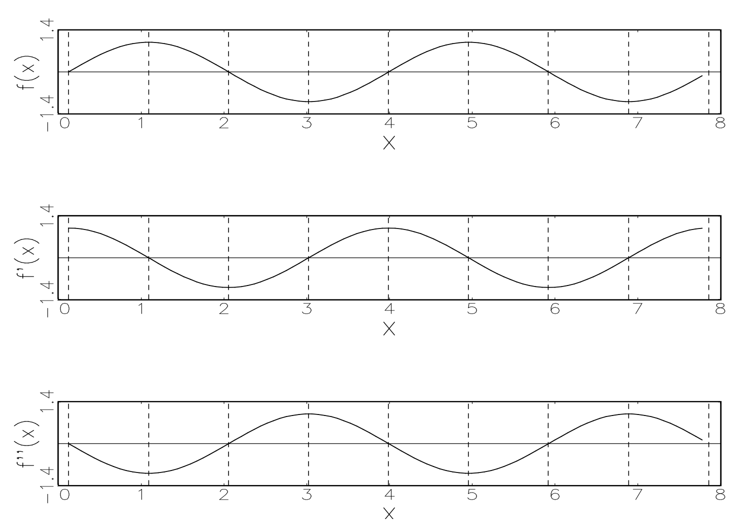
   (a) If $m = e^{\,p}$, then $p = \ln m$:

   The log of $m$ is $p$.

   (b) Or,

$$\ln m = p \ \text{ which is equivalent to } \ e^p = m$$

   (c) Which looks like:



4. * Log of Products

   (a) Let

   $$\begin{aligned} m &= e^p \Longleftrightarrow \ln m = p \\ n &= e^q \Longleftrightarrow \ln n = q \end{aligned}$$

   (b) Then, multiply $m$ times $n$:

   $$\begin{aligned} m \times n &= e^p \times e^q \\ &= e^{(p+q)} \end{aligned}$$

   (c) Taking the log of both sides:

   $$\begin{aligned} \ln(m \times n) &= \ln\left[e^{(p+q)}\right] \\ &= p + q \\ &= \ln m + \ln n \end{aligned}$$

   (d) For example:

   $$\begin{aligned} 2 \times 3 &= 6 \\ \ln(2 \times 3) &= \ln 2 + \ln 3 = \\ &= 0.69315... + 1.0986... \\ &= 1.7918... \\ &= \ln 6 \end{aligned}$$

5. * Log of Quotients

   $$\begin{aligned} \ln\left(\frac{m}{n}\right) &= \ln m - \ln n \\ \ln\left(\frac{3}{2}\right) &= .40547 = \ln 3 - \ln 2 = 1.0986 - .69315 \end{aligned}$$

   The logit: $\ln\left(\dfrac{p}{1-p}\right) =$

6. Inverse operations

   (a) $\ln(k)$ is that power of $e$ that equals $k$:

   $$k = e^{\ln k}$$

   (b) $\ln(e^k)$ is that power of $e$ that equals $e^k$, namely $k$:

   $$\ln e^k = k$$

(c) and

$$e^{\ln k} = k$$

7. Log of Power

$$\begin{aligned} \ln m^n &= n \ln m \\ \ln 3^2 &= \ln 9 = 2.1972 = 2 \ln 3 = 2\,(1.0986) \end{aligned}$$

8. Example from Regression

(a) Assume that

$$y = \alpha x_1^{\beta_1} x_2^{\beta_2} \varepsilon$$

(b) Taking logs:

$$\begin{aligned} \ln y &= \ln \left( \alpha x_1^{\beta_1} x_2^{\beta_2} \varepsilon \right) \\ &= \ln \alpha + \ln x_1^{\beta_1} + \ln x_2^{\beta_2} + \ln \varepsilon \\ &= \ln \alpha + \beta_1 \ln x_1 + \beta_2 \ln x_2 + \varepsilon^* \\ &= \alpha^* + \beta_1 x_1^* + \beta_2 x_2^* + \varepsilon^* \end{aligned}$$

## A.6 Vector Algebra

1. Consider the regression equation for observation $i$:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

Vector multiplication allows us to write this more simply.

2. For example, let $\boldsymbol{\beta}' = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix}$, then

$$\mathbf{x}\boldsymbol{\beta} = \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

3. More generally, consider $\boldsymbol{\beta}_{K\times 1}$ and $\mathbf{x}_{1\times K}$, then by definition:

$$\mathbf{x}\boldsymbol{\beta} = \beta_0 + \sum_{i=1}^{K} \beta_i x_i = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \cdots$$

## A.7 Probability Distributions

▶ Let $X$ be a random variable with discrete outcomes $x$. The frequency of those outcomes is the *probability distribution:*

$$f(x) = \Pr(X = x)$$

**Bernoulli Distribution** For example, let $y$ indicate the outcome of a fair coin. Then, $y = 0$ or 1, and

$$\Pr(y = 0) = .4 \quad \text{and} \quad \Pr(y = 1) = .6$$

▶ For all probability distributions:

1. All probabilities are between zero and one: $0 \leq f(x) \leq 1$
2. Probabilities sum to one: $\sum_x f(x_i) = 1$

▶ For a continuous random variable, $f(x)$ is called a *probability density function* or *pdf*.

1. $f(x) = 0$. Why? Pick any two numbers. Can you find a number in between them?
2. $\Pr(a \leq x \leq b) = \int_a^b f(x)dx \geq 0$
3. $\int_{-\infty}^{\infty} f(t)dt = 1$

**Normal Distribution**

1. The pdf mean $\mu$ and standard deviation $\sigma$, $x \sim \mathcal{N}(\mu, \sigma^2)$ , is:

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \tag{A.11}$$

2. This defines the classic bell curve:

3. If $\mu = 0$ and $\sigma^2 = 1$:

$$\phi(x) = f(x \mid 0, 1) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \tag{A.12}$$

### A.7.1 Cumulative Distribution Function (cdf)

▶ The *cdf* is the probability of a value up to or equal to a specific value.

- For discrete random variables: $F(x) = \sum_{X \leq x} f(x) = \Pr(X \leq x)$
- For a continuous variable: $F(x) = \displaystyle\int_{-\infty}^{x} f(t)dt = \Pr(X \leq x)$

▶ For the cdf:

1. $0 \leq F(x) \leq 1$ .
2. If $x > y$, then $F(x) \geq F(y)$ .
3. $F(-\infty) = 0$ and $F(\infty) = 1$ .

### A.7.2 * Computing the Area Within a Distribution

▶ Consider the distribution $f(x)$, where $F(x) = \Pr(X \leq x)$:

$$\Pr(a \leq X \leq b) = \Pr(X \leq b) - \Pr(X \leq a) = F(b) - F(a)$$

### A.7.3 * Expectation

▶ The mean of $N$ sample values of $X$ is:

$$\bar{x} = \frac{\sum_{i=1}^{N} X_i}{N}$$

For example:

$$\frac{1+1+4+10}{4} = \left(1 \times \frac{2}{4}\right) + \left(4 \times \frac{1}{4}\right) + \left(10 \times \frac{1}{4}\right) = 4$$

▶ The *expectation* is defined in terms of the population:

- For discrete variables:

$$\mathrm{E}(X) = \sum_x f(x)x = \sum_x \Pr(X = x)x$$

- For continuous variables:

$$\mathrm{E}(X) = \int_x f(x)x\,dx$$

*** Example of Expectation of Binary Variable** If $X$ has values 0 and 1 with probabilities $\frac{1}{4}$ and $\frac{3}{4}$, then

$$\begin{aligned}\mathrm{E}(X) &= \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{3}{4}\right) = \frac{3}{4} = \Pr(x = 1) \\ &= [\text{Value}_1 \Pr(\text{Value}_1)] + [\text{Value}_2 \Pr(\text{Value}_2)]\end{aligned}$$

*** Expectation of Sums**

▶ If $X$ and $Y$ are random variables, and $a$, $b$ and $c$ are constants, then

$$\mathrm{E}(a + bX + cY) = a + b\mathrm{E}(X) + c\mathrm{E}(Y) \tag{A.13}$$

▶ Example: Let

$$y_i = \alpha + \sum_{k=1}^{K} \beta_k x_{ki} + \varepsilon_i$$

Then

$$\begin{aligned}
\mathrm{E}(y_i) &= \mathrm{E}\left(\alpha + \sum_{k=1}^{K} \beta_k x_{ki} + \varepsilon_i\right) \\
&= \mathrm{E}(\alpha) + \mathrm{E}\left(\sum_{k=1}^{K} \beta_k x_{ki}\right) + \mathrm{E}(\varepsilon_i) \\
&= \alpha + \sum_{k=1}^{K} \beta_k \mathrm{E}(x_{ik})
\end{aligned}$$

**Conditional Expectations**

▶ Conditioning means holding some things constant while something else changes.

▶ Example: Let \$ be income.

- E(\$) tells us the mean \$, but is not useful for telling us how other variables affect \$.
- Let $S$ be the sex of the respondent. We might compute:

$$\mathrm{E}(\$ \mid S = \text{female}) = \text{Expected \$ for females}$$

- This allows us to see how the expectation varies by the level of other variables.

▶ Example: If $y = \mathbf{x}\boldsymbol{\beta} + \varepsilon$, then

$$\mathrm{E}(y \mid \mathbf{x}) = \mathrm{E}(\mathbf{x}\boldsymbol{\beta} + \varepsilon) = \mathrm{E}(\mathbf{x}\boldsymbol{\beta}) + \mathrm{E}(\varepsilon) = \mathbf{x}\boldsymbol{\beta}$$

### A.7.4 The Variance

▶ The variance is defined as

$$s^2 = \frac{\sum_{i=1}^{N}(x_i - \overline{x})^2}{N}$$

▶ Variance for a Population: Let $f(x) = \Pr(X = x)$.

- If $x$ is discrete:

$$\mathrm{Var}(X) = \sum_x \left[x - \mathrm{E}(x)\right]^2 f(x) \tag{A.14}$$

- If $x$ is continuous:

$$\mathrm{Var}(X) = \int_x \left[x - \mathrm{E}(x)\right]^2 f(x)dx \tag{A.15}$$

**Example of Variance of Binary Variable** If $X$ has values 0 and 1 with probabilities $\frac{1}{4}$ and $\frac{3}{4}$, then $\mathrm{E}(X) = \frac{3}{4}$, and

$$\begin{aligned} \mathrm{Var}(X) &= \left(\left[0 - \frac{3}{4}\right]^2 \times \frac{1}{4}\right) + \left(\left[1 - \frac{3}{4}\right]^2 \times \frac{3}{4}\right) \\ &= \left(\frac{9}{16} \times \frac{1}{4}\right) + \left(\frac{1}{16} \times \frac{3}{4}\right) = \frac{9}{64} + \frac{3}{64} = \frac{12}{64} = \frac{3}{16} \\ &= \mathrm{E}\left(X\right)\left[1 - \mathrm{E}\left(X\right)\right] \end{aligned}$$

*** Variance of a Linear Transformation**

▶ Let $X$ be a random variable, and $a$ and $b$ be constants. Then,

$$\mathrm{Var}(a + bX) = b^2 \mathrm{Var}(X) \tag{A.16}$$

*** Variance of a Sum**

▶ Let $X$ and $Y$ be two random variables with constants $a$ and $b$:

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X, Y) \tag{A.17}$$

▶ Let $Y = \sum_{i=1}^{K} X_i$. If the $X$'s are uncorrelated, then

$$\mathrm{Var}(Y) = \mathrm{Var}(\sum_{i=1}^{K} X_i) = \sum_{i=1}^{K} \mathrm{Var}(X_i) \tag{A.18}$$

## A.8 **Rescaling Variables

Often we want to use addition and multiplication to change a variable with mean $\mu$ and variance $\sigma^2$ into a variable with mean 0 and variance 1. This is called rescaling.

1. Consider $X$ where
$$\mathrm{E}\left(x\right) = \mu \quad \text{and} \quad \mathrm{Var}\left(x\right) = \sigma^2$$

2. By subtracting the mean, the expectation becomes zero:
$$\mathrm{E}\left(x-\mu\right) = \mathrm{E}\left(x\right) - \mathrm{E}\left(\mu\right) = \mu - \mu = 0$$

3. But the variance is unchanged:
$$\mathrm{Var}\left(x-\mu\right) = \mathrm{Var}\left(x\right) = \sigma^2$$

4. Dividing by $\sigma$:
$$\mathrm{E}\left(\frac{x}{\sigma}\right) = \mathrm{E}\left(\frac{1}{\sigma}x\right) = \frac{1}{\sigma}\mathrm{E}\left(x\right) = \frac{1}{\sigma}\mu = \frac{\mu}{\sigma}$$

5. Subtracting $\mu$ and dividing by $\sigma$ does not change the mean:
$$\mathrm{E}\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\mathrm{E}\left(x-\mu\right) = \frac{1}{\sigma}0 = 0 \tag{A.19}$$

6. But, the variance becomes one:
$$\mathrm{Var}\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\mathrm{Var}\left(x-\mu\right) = \frac{1}{\sigma^2}\mathrm{Var}\left(x\right) = 1 \tag{A.20}$$

### Stata: Standardizing Variables

```
. use science2, clear
. sum pub9

Variable |     Obs        Mean   Std. Dev.       Min        Max
---------+-----------------------------------------------------
pub9     |     308    4.512987   5.315134          0         33

. gen p9_mn = pub9 - r(mean)
. sum p9_mn

Variable |     Obs        Mean   Std. Dev.       Min        Max
---------+-----------------------------------------------------
p9_mn    |     308   -5.71e-09   5.315134  -4.512987   28.48701

. gen p9_sd = p9_mn/5.315134
. gen p9_sd2 = (pub9 - 4.512987)/5.315134
. egen p9_sdez = std(pub9)
. sum p9_sd p9_sd2 p9_sdez

Variable |     Obs        Mean   Std. Dev.       Min        Max
---------+-----------------------------------------------------
p9_sd    |     308    4.02e-09   .9999999  -.8490825   5.359604
p9_sd2   |     308    2.18e-10   .9999999  -.8490825   5.359604
p9_sdez  |     308   -6.77e-10          1  -.8490825   5.359604
```

# A.9 Distributions

## A.9.1 Bernoulli

▶ $X$ has a Bernoulli distribution if it has two possible outcomes:

$$\Pr(X = 1) = p \qquad \text{and} \qquad \Pr(X = 0) = 1 - p$$

▶ Then:

$$f(x \mid p) = p^x (1-p)^{1-x} = \Pr(X = x \mid p)$$

▶ That is:

$$f(0 \mid p) = p^0 (1-p)^1 = 1 - p \qquad \text{and} \qquad f(1 \mid p) = p^1 (1-p)^0 = p$$

▶ It can be shown that

$$\mathrm{E}(X) = p \qquad \text{and} \qquad \mathrm{Var}\,(X) = p\,(1-p) \tag{A.21}$$

▶ Note how the variance is related to the mean:

| $\mathrm{E}(X) = p$ | $\mathrm{Var}\,(X) = p\,(1-p)$ |
|---|---|
| 0.1 | 0.090 |
| 0.2 | 0.160 |
| 0.3 | 0.210 |
| 0.4 | 0.240 |
| 0.5 | 0.250 |
| 0.6 | 0.240 |
| 0.7 | 0.210 |
| 0.8 | 0.160 |
| 0.9 | 0.090 |

### A.9.2 Normal

▶ The pdf for a normal distribution with mean $\mu$ and standard deviation $\sigma$ is

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \tag{A.22}$$

▶ If $x$ is distributed normally with mean $\mu$ and standard deviation $\sigma$:

$$x \sim \mathcal{N}(\mu, \sigma^2)$$

▶ The cdf is defined as

$$F(x \mid \mu, \sigma) = \int_{-\infty}^{x} f(t \mid \mu, \sigma) dt$$

**Standardized Normal**

▶ If $x \sim \mathcal{N}(0,1)$, we define:

$$\text{pdf: } \phi(x) = \tfrac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \qquad \text{cdf: } \Phi(x) = \textstyle\int_{-\infty}^{x} \phi(t)dt$$

▶ You can move from an unstandardized to a standardized normal distribution.

- Let $x \sim \mathcal{N}(0, \sigma^2)$
- Then,

$$\begin{aligned} f(x \mid 0, \sigma) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right) \\ &= \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} \exp\left(-\left(\frac{x}{\sigma}\right)^2/2\right) \\ &= \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right) \end{aligned} \tag{A.23}$$

**Area Under the Curve**

▶ If $x \sim \mathcal{N}(0,1)$, then

$$\Pr(a \le x \le b) = \Phi(b) - \Phi(a) \tag{A.24}$$

**Linear Transformation of a Normal**

▶ If

$$x \sim \mathcal{N}\left(\mu, \sigma^2\right)$$

▶ Then

$$a + bx \sim \mathcal{N}\left(a + b\mu, b^2\sigma^2\right) \tag{A.25}$$

**Sums of Normals**

▶ Let $\text{Cor}(x_1, x_2) = \rho$, where

$$x_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right) \qquad \text{and} \qquad x_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$$

▶ Then

$$\alpha_1 x_1 + \alpha_2 x_2 \sim \mathcal{N}(\left[\alpha_1\mu_1 + \alpha_2\mu_2\right], \left[\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + 2\rho\alpha_1\alpha_2\sigma_1\sigma_2\right]) \tag{A.26}$$

▶ When $\rho = 0$:

$$\alpha_1 x_1 + \alpha_2 x_2 \sim \mathcal{N}(\left[\alpha_1\mu_1 + \alpha_2\mu_2\right], \left[\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2\right])$$

### A.9.3 Chi-square

▶ Let $\phi_i$ $(i = 1 \text{ to } df)$ be *independent*, standard normal variates.

▶ *Define*:

$$X^2_{df} \equiv \sum_{i=1}^{df} \phi_i^2 \sim \chi^2_{df}$$

▶ The chi-square distribution is defined as the sum of independent squared normal variables.

▶ The mean and variance:

$$\mathrm{E}\left(X^2_{df}\right) = df \qquad \text{and} \qquad \mathrm{Var}\left(X^2_{df}\right) = 2df$$

#### Adding Chi-squares

- Let $x \sim \chi^2_{df_x}$ and $y \sim \chi^2_{df_y}$
- If $x$ and $y$ are independent:

$$x + y \sim \chi^2_{df_x + df_y}$$

#### Shape

- With 1 *df*, the distribution is highly skewed.
- As $df \to \infty$, the chi-square becomes distributed normally.

**Question from Intro to Statistics** Consider the chi-square test in contingency tables:

$$X^2 = \sum_{\text{all cells}} \frac{(\text{obs} - \exp)^2}{\exp} \sim X^2_{df} \quad \text{with } df = (\#\text{rows} - 1)(\#\text{columns} - 1)$$

- Why would this be distributed as chi-square? Why those degrees of freedom?

### A.9.4 *F*-distribution

▶ Let $X_1$ and $X_2$ be independent chi-square variables with degrees of freedom $r_1$ and $r_2$.

▶ The $F$-distribution is *defined* as:

$$F_{r_1, r_2} \equiv \frac{X_1/r_1}{X_2/r_2}$$

### A.9.5 *t*-distribution

▶ Consider $z \sim \phi$ and $x \sim \chi_{df}$ , where $z$ and $x$ are independent.

▶ Then the $t$-distribution with $df$ degrees of freedom is *defined* as:

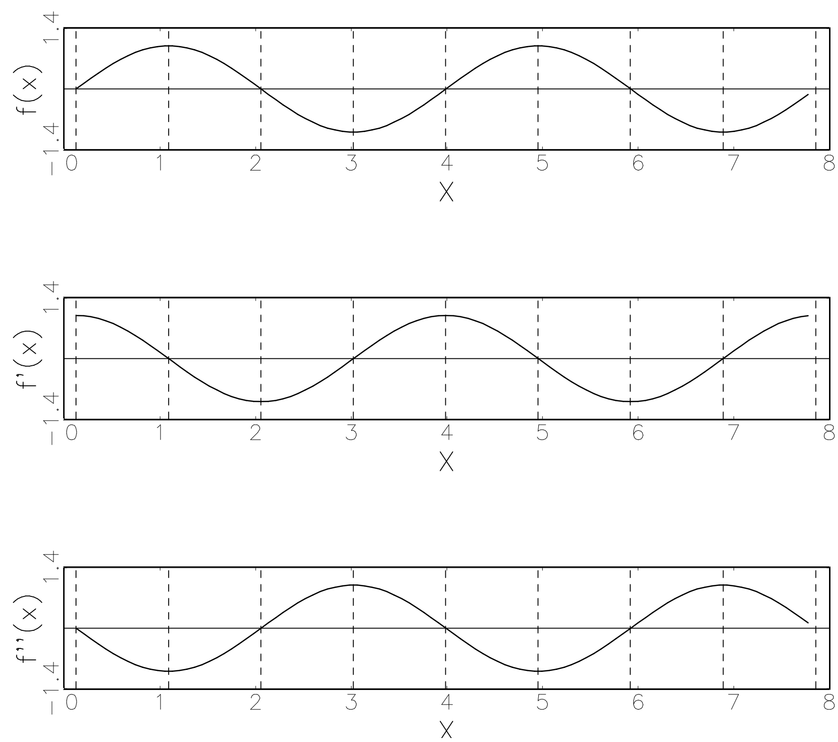
$$t_{df} \equiv \frac{z}{\sqrt{x/df}}$$

### A.9.6 Relationships among normal, t, chi-square and F

1. $z = t_\infty$
2. $z^2 = X_1^2 = F_{1,\infty} = t_\infty^2$
3. $t_{df}^2 = F_{1,df}$
4. $\dfrac{X_{df}^2}{df} = F_{r,\infty}$

## A.10 Calculus

▶ The two central ideas in calculus are the derivative and the integral.




▶ **Derivative**: The derivative is the slope of a curve $y = f(x)$:

$$\frac{dy}{dx} = f'(x) \tag{A.27}$$

▶ The second derivative indicates how quickly the slope of the curve is changing:

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = f''(x) \tag{A.28}$$

▶ If the curve is defined as $y = f(x, z)$, we write the *partial derivative* with respect to $x$ as

$$\frac{\partial f(x,z)}{\partial x} \tag{A.29}$$

- Imagine half of a hard boiled egg setting on a table; slice it from the top to the table.
- The partial derivative is the slope on the resulting curve.

▶ **Integral**: The integral is the area under a curve.

▶ For example, if a curve is defined as $y = f(x)$, the area under the curve from point $a$ to point $b$ is computed with the integral:

$$\int_b^a f(t)\,dt$$

# A.11 Matrix Algebra

## A.11.1 Basic Definitions

**Matrix** is an array of numbers, arranged in *rows* and *columns:*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

## A.11.2 Transposing a Matrix

▶ The transpose is indicated by the prime or superscript T. For example: $\mathbf{A}'$ or $\mathbf{A}^{\mathrm{T}}$.

$$\text{If } \mathbf{A} = \begin{bmatrix} 11 & 12 \\ 21 & 22 \end{bmatrix}, \text{ then } \mathbf{A}' = \begin{bmatrix} 11 & 21 \\ 12 & 22 \end{bmatrix}$$

▶ Transposing the Transpose

$$\mathbf{A}'' = \mathbf{A} \tag{A.30}$$

▶ If $A$ is a symmetric matrix, then $\mathbf{A}' = \mathbf{A}$

## A.11.3 Addition and Subtraction

▶ Addition:

$$\mathbf{A} + \mathbf{B} = \{a_{rc} + b_{rc}\}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 7 & 11 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+3 \\ 4+7 & 5+11 \end{bmatrix}$$

▶ Transposes of added matrices:

$$\begin{aligned} (\mathbf{A} + \mathbf{B})' &= \mathbf{A}' + \mathbf{B}' \qquad \text{(A.31)} \\ \left( \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 7 & 11 \end{bmatrix} \right)' &= \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}' + \begin{bmatrix} 1 & 3 \\ 7 & 11 \end{bmatrix}' \\ &= \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ 3 & 11 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ 5 & 16 \end{bmatrix} \end{aligned}$$

▶ Subtraction of matrices:

$$\mathbf{A} - \mathbf{B} = \{a_{rc} - b_{rc}\}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 7 & 11 \end{bmatrix} = \begin{bmatrix} 1-1 & 2-3 \\ 4-7 & 5-11 \end{bmatrix}$$

## A.11.4 Scalar Multiplication

$$\alpha \mathbf{A} = \alpha \{a_{rc}\} = \{\alpha \times a_{rc}\}$$

$$3 \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 \times 1 & 3 \times 2 \\ 3 \times 4 & 3 \times 5 \end{bmatrix}$$

### A.11.5 Matrix Multiplication

**Vector** is a matrix with one dimension equal to one.

▶ A *column vector* is an $R \times 1$ matrix:

$$\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{or} \quad \mathbf{c}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

▶ A *row vector* is a $1 \times C$ matrix:

$$\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

* **Vector Multiplication** Consider $\boldsymbol{\beta}_{K\times 1}$ and $\mathbf{x}_{1\times K}$, then by definition:

$$\mathbf{x}\boldsymbol{\beta} = \sum_{i=1}^{3} \beta_i x_i = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

▶ For example, let $\boldsymbol{\beta}' = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix}$, then

$$\mathbf{x}\boldsymbol{\beta} = \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

**Matrix Multiplication** For $\mathbf{A}_{R\times K}$ and $\mathbf{B}_{K\times C}$, the *matrix product* $\mathbf{C}_{R\times C} = \mathbf{AB}$ equals:

$$\{c_{rc}\} = \left\{ \sum_{i=1}^{K} a_{ri} b_{ic} \right\}$$

▶ Note that element $c_{rc}$ is the vector multiplication of row $r$ from $\mathbf{A}$ and column $c$ from $\mathbf{B}$.

▶ Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1a + 2d + 3g & 1b + 2e + 3h & 1c + 2f + 3i \\ 4a + 5d + 6g & 4b + 5e + 6h & 4c + 5f + 6i \end{bmatrix}$$

▶ Example from Regression:

$$\begin{aligned} \mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \\ \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} &= \begin{bmatrix} 1 & x_{11} & x_{12} \\ \vdots & \vdots & \vdots \\ 1 & x_{N1} & x_{N2} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{bmatrix} \\ &= \begin{bmatrix} \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \varepsilon_1 \\ \vdots \\ \beta_0 + \beta_1 x_{N1} + \beta_2 x_{N2} + \varepsilon_N \end{bmatrix} \end{aligned}$$

### A.11.6 Inverse

▶ An *identity matrix* is a square matrix with 1's on the diagonal, and 0's elsewhere.

▶ If $\mathbf{A}$ is square, then $\mathbf{A}^{-1}$ is the *inverse* of $\mathbf{A}$ if and only if

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{A.32}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -1\frac{2}{3} & \frac{2}{3} \\ 1\frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1. If $\mathbf{A}^{-1}$exists, it is unique.
2. If $\mathbf{A}^{-1}$does not exist, $\mathbf{A}$ is called *singular*.

### A.11.7 Rank

▶ *Rank* is the size of the largest submatrix that can be inverted.

▶ A matrix is of *full rank* if the rank is equal to the minimum of the number of rows and columns.

▶ Problems occur in estimation when a matrix is encountered that is not of full rank.

▶ When this occurs, messages such as the following are generated:

- Matrix is not of full rank.
- Singular matrix encountered.
- Matrix cannot be inverted.
- An inverse does not exist.